

New Recursion Relations for Tree Amplitudes of Gluons

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We present new recursion relations for tree amplitudes in gauge theory that give very compact formulas. Our relations give any tree amplitude as a sum over terms constructed from products of two amplitudes of fewer particles multiplied by a Feynman propagator. The two amplitudes in each term are physical, in the sense that all particles are on-shell and momentum conservation is preserved. This is striking, since it is just like adding certain factorization limits of the original amplitude to build up the full answer. As examples, we recompute all known tree-level amplitudes of up to seven gluons and show that our recursion relations naturally give their most compact forms. We give a new result for an eight-gluon amplitude, $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-)$. We show how to build any amplitude in terms of three-gluon amplitudes only.

December 2004

1. Introduction

Scattering amplitudes of gluons possess a remarkable simplicity that is not manifest by their computation using Feynman diagrams.

At tree-level the first hints of this hidden simplicity were first unveiled by the work of Parke and Taylor [1]. They conjectured a very simple formula for all amplitudes with at most two negative helicity gluons,¹

$$\begin{aligned} A^{\text{tree}}(1^+, 2^+, \dots, n^+) &= 0, & A^{\text{tree}}(1^-, 2^+, \dots, n^+) &= 0, \\ A^{\text{tree}}(1^-, 2^+, \dots, j^-, \dots, n^+) &= \frac{\langle 1 j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n-1 n \rangle \langle n 1 \rangle}. \end{aligned} \tag{1.1}$$

These formulas were proven by Berends and Giele using their recursion relations [2]. Many more analytic formulas were obtained the same way [3,4,5,6].

Even though these formulas were much simpler than expected, their form is not as simple as the Parke-Taylor amplitudes.

In a remarkable work [7], Witten discovered that when tree-level amplitudes are transformed into twistor space [8] all of them have a simple geometrical description. This led to the introduction of MHV diagrams (also known as the CSW construction) in [9], where all tree amplitudes are computed by sewing MHV amplitudes continued off-shell with Feynman propagators, as well as to a computation using connected instantons that reduces the problem of finding tree amplitudes to that of solving certain algebraic equations [10]. Much progress has been made in the past year [11].

At one-loop, the situation is much more complicated. This is clear from the fact that the state of the art in QCD is only five-gluon amplitudes. However, the situation is much better for supersymmetric amplitudes. Another motivation for studying supersymmetric amplitudes is that they are useful in the computation of QCD amplitudes (for a review see [12]). The reason supersymmetric amplitudes are simpler is because they are four-dimensional cut constructible [13,14]. In particular, $\mathcal{N} = 4$ amplitudes have the simplest structure as they can be written as linear combinations of scalar box integrals with rational coefficients. Since the integrals are known explicitly, computing one-loop amplitudes is reduced to the problem of finding the coefficients.

One-loop $\mathcal{N} = 4$ amplitudes are UV finite but IR divergent. The IR behavior of all one-loop $\mathcal{N} = 4$ amplitudes is universal and well understood [15]. It relates some

¹ Here we are suppressing a trace factor, a delta function of momentum conservation and powers of the coupling constant.

linear combination of the coefficients to the tree-level contribution of the amplitude being computed.

These IR equations are usually used as consistency checks of the coefficients and in many cases as a way of obtaining hard-to-compute coefficients in terms of other coefficients. Once the coefficients are computed, they can be used to give new representations of tree-level amplitudes [16,17,18].

However, the situation has changed. In [19], we introduced a new method for computing all coefficients in $\mathcal{N} = 4$ one-loop amplitudes in a simple and systematic manner. Roughly speaking, every coefficient is given as the product of four tree-level amplitudes with fewer external legs than the amplitude being computed.

This leads to a surprising new application of the IR equations. They now become new recursion relations for tree-level amplitudes!

A particularly simple linear combination of IR equations was found in [18],

$$A_n^{\text{tree}} = \frac{1}{2} \sum_{i=1}^{n-3} B_{1,i+1,n-1,n} \quad (1.2)$$

where B_{abcd} denotes the coefficients of a scalar box function with momenta $K_1 = p_a + p_{a+1} + \dots + p_{b-1}$, $K_2 = p_b + \dots + p_{c-1}$, and so on. In (1.2) only two-mass-hard box function coefficients enter (for $i = 1$ and $i = n - 3$, the two-mass-hard becomes a one-mass function).

From the result of [19], each coefficient B in (1.2) can be computed as a sum of products of four tree-level amplitudes of fewer external gluons. The sum runs over all possible particles of the $\mathcal{N} = 4$ multiplet and helicity configurations in the loop.

It turns out that one can do better. In this paper, we propose a new recursion relation for tree-level amplitudes that involves a sum over terms built from the product of two tree-level amplitudes times a Feynman propagator. Schematically, it is given by

$$A_n = \sum_{i=1}^{n-3} \sum_{h=\pm} A_{i+2}^h \frac{1}{P_{n,i}^2} A_{n-i}^{-h} \quad (1.3)$$

where A_k denotes a certain k -gluon tree-level amplitude, and $P_{n,i}$ is the sum of the momenta of gluons $n, 1, 2, \dots, i$. The index h labels the two possible helicity configurations of the particle being “exchanged” between two amplitudes.

Note that the form of (1.3) is quite striking because each term in the i sum is identical to the factorization limit of A_n in the $P_{n,i}$ channel. More explicitly, it is well known that the most leading singular piece of A_n in the kinematical regime close to $P_{n,i}^2 \rightarrow 0$ is

$$A_n|_{P_{n,i}^2 \rightarrow 0} = \sum_{h=\pm} A_{i+2}^h \frac{1}{P_{n,i}^2} A_{n-i}^{-h}. \quad (1.4)$$

This is known as a multiparticle singularity (for a review see [12]).

Note that in (1.4), both tree-amplitudes are on-shell and momentum conservation is preserved. This means that each tree amplitude becomes a physical amplitude.

Very surprisingly, it turns out that the tree amplitudes in (1.3) also are on-shell and momentum conservation is preserved.

In order to test our formula (1.3), we recomputed all tree amplitudes up to seven gluons and found complete agreement with the results in the literature. It is worth mentioning that in [16], formulas for next-to-MHV seven gluon amplitudes were presented that are simpler than any previously known form in the literature. Via collinear limits, a very compact formula for $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ was given in [18]. Also in [18], a very compact formula for the amplitude $A(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$ was presented. It turns out that a straightforward use of our formula (1.3) gives rise to the same simple and compact formulas. We also give similar formulas for all other six-gluon amplitudes.

As a new result we present the eight-gluon amplitude with alternate helicity configuration, i.e. $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-, 8^+)$.² We describe how repeated applications of the recursion relations will reduce any amplitude to a product of three-gluon amplitudes and propagators. This is very surprising, given that the Yang-Mills Lagrangian has cubic and quartic interactions. We also discuss an interesting set of amplitudes that are closed under the recursion relations and speculate on the possibility of solving for them explicitly.

This paper is organized as follows: In section 2, we present the recursion relation in detail. We illustrate how to use it in practice by giving a detailed calculation of $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. In section 3, we present the results obtained from our recursion relations applied to all amplitudes of up to seven gluons and a particular eight-gluon case. In section 4, we present our result for the amplitude $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-, 8^+)$. In section 5, we discuss some interesting directions for the future. We give an outline of

² It would also be possible to derive this amplitude from a suitable limit of the NNMHV amplitudes presented in [20].

a possible proof of our recursion relations and suggest a way to prove the crucial missing step. We discuss its possible relation to MHV diagrams (the CSW construction), and point out a class of amplitudes closed under the recursion relations. Finally, in the appendix, we give some details on the calculations involved in the outline of a possible proof given in section 5.

Throughout the paper, we use the following notation and conventions along with those of [7]. The external gluon labeled by i carries momentum p_i .

$$\begin{aligned}
t_i^{[r]} &\equiv (p_i + p_{i+1} + \dots + p_{i+r-1})^2 \\
\langle i | \sum_r p_r | j \rangle &\equiv \sum_r \langle i \ r \rangle [r \ j] \\
\langle i | (\sum_r p_r) (\sum_s p_s) | j \rangle &\equiv \sum_r \sum_s \langle i \ r \rangle [r \ s] \langle s \ j \rangle \\
[i | (\sum_r p_r) (\sum_s p_s) | j \rangle &\equiv \sum_r \sum_s [i \ r] \langle r \ s \rangle [s \ j] \\
\langle i | (\sum_r p_r) (\sum_s p_s) (\sum_t p_t) | j \rangle &\equiv \sum_r \sum_s \sum_t \langle i \ r \rangle [r \ s] \langle s \ t \rangle [t \ j]
\end{aligned} \tag{1.5}$$

2. Recursion Relations

Consider any n -gluon tree-level amplitude with any helicity configuration. Without loss of generality let us take the labels of the gluons such that the $(n-1)$ -th gluon has negative helicity and the n -th gluon has positive helicity.³ Then we claim that the following recursion relation for tree amplitudes is valid:

$$\begin{aligned}
A_n(1, 2, \dots, (n-1)^-, n^+) = \\
\sum_{i=1}^{n-3} \sum_{h=+,-} \left(A_{i+2}(\widehat{n}, 1, 2, \dots, i, -\widehat{P}_{n,i}^h) \frac{1}{P_{n,i}^2} A_{n-i}(\widehat{P}_{n,i}^{-h}, i+1, \dots, n-2, \widehat{n-1}) \right), \tag{2.1}
\end{aligned}$$

where

$$\begin{aligned}
P_{n,i} &= p_n + p_1 + \dots + p_i, \\
\widehat{P}_{n,i} &= P_{n,i} + \frac{P_{n,i}^2}{\langle n-1 | P_{n,i} | n \rangle} \lambda_{n-1} \widetilde{\lambda}_n, \\
\widehat{p}_{n-1} &= p_{n-1} - \frac{P_{n,i}^2}{\langle n-1 | P_{n,i} | n \rangle} \lambda_{n-1} \widetilde{\lambda}_n, \\
\widehat{p}_n &= p_n + \frac{P_{n,i}^2}{\langle n-1 | P_{n,i} | n \rangle} \lambda_{n-1} \widetilde{\lambda}_n.
\end{aligned} \tag{2.2}$$

³ Recall that amplitudes with all positive or all negative helicity gluons vanish [1,2].

At this point we should make some comments about this formula. First, note that the momenta $P_{n,i}$, p_{n-1} and p_n are all shifted in the same way. The term we add,

$$\frac{P_{n,i}^2}{\langle n-1|P_{n,i}|n\rangle}\lambda_{n-1}\tilde{\lambda}_n, \quad (2.3)$$

seems peculiar at this point, but it arises naturally from the discussion in section 5. The shift (2.3) is not parity invariant. Moreover, it cannot be interpreted as a vector in Minkowski space, since $\tilde{\lambda}$ and λ are independent, but it has a natural meaning in $(--++)$ signature, as we discuss in section 5.

Note that each tree-level amplitude in (2.1) has all external gluons on-shell. Indeed, it is easy to see that $\hat{P}_{n,i}^2 = \hat{p}_n^2 = \hat{p}_{n-1}^2 = 0$. It is interesting to note that $\hat{P}_{n,i}$ is a generalization of the formula used in [21,22] to define non-MHV amplitudes off-shell. In contrast, here we used it in order to keep the amplitudes on-shell while momentum conservation is preserved.

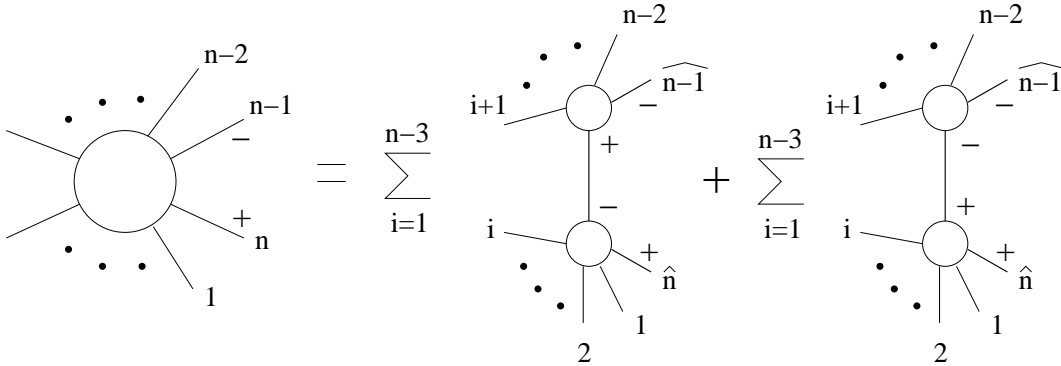


Fig. 1: Pictorial representation of the recursion relation (2.1). Note that the difference between the terms in the two sums is just the helicity assignment of the internal line.

The fact that momentum conservation is preserved in each of the tree-level amplitudes, as anticipated in the introduction, might be a little puzzling at first. Consider in particular the limiting cases in the sum (2.1), i.e. $i = 1$ and $i = n - 3$. For these two values of i , one tree amplitude is a three-gluon amplitude on-shell, and it is well known that this vanishes in Minkowski signature $(-+++)$. Here, on the other hand, the momenta are taken in $(--++)$ signature, and three-gluon amplitudes on-shell are non-trivial.

2.1. Tips for Using the Recursion Relation

Let us explain the way we use (2.1) and (2.2) in practice. First note that from the definition of \widehat{p}_{n-1} and \widehat{p}_n one can read off their spinor components very easily. Recall that $p_{n-1} = \lambda_{n-1}\widetilde{\lambda}_{n-1}$ and $p_n = \lambda_n\widetilde{\lambda}_n$, therefore

$$\begin{aligned}\widehat{\lambda_{n-1}} &= \lambda_{n-1}, \\ \widetilde{\widehat{\lambda_{n-1}}} &= \widetilde{\lambda_{n-1}} - \frac{P_{n,i}^2}{\langle n-1|P_{n,i}|n\rangle}\widetilde{\lambda}_n, \\ \widehat{\lambda_n} &= \lambda_n + \frac{P_{n,i}^2}{\langle n-1|P_{n,i}|n\rangle}\lambda_{n-1}, \\ \widetilde{\widehat{\lambda_n}} &= \widetilde{\lambda}_n.\end{aligned}\tag{2.4}$$

Finally, we use the following identities to compute any spinor product involving $\widehat{P}_{n,i}$.

$$\begin{aligned}\langle \bullet | \widehat{P}_{n,i} \rangle &= -\langle \bullet | P_{n,i} | n \rangle \times \frac{1}{\omega} \\ [\widehat{P}_{n,i} | \bullet] &= -\langle n-1 | P_{n,i} | \bullet \rangle \times \frac{1}{\overline{\omega}}\end{aligned}\tag{2.5}$$

Here $\omega = [\widehat{P}_{n,i} | n]$ and $\overline{\omega} = \langle n-1 | \widehat{P}_{n,i} \rangle$. Since the $\widehat{P}_{n,i}$ have opposite helicities on both amplitudes, the product $A_{i+2}A_{n-i}$ must have degree zero under the rescaling of $\lambda_P \rightarrow t\lambda_P$ and $\widetilde{\lambda}_P \rightarrow t^{-1}\widetilde{\lambda}_P$. Therefore, the factors ω and $\overline{\omega}$ can only show up in the final answer in the invariant combination $\omega\overline{\omega}$. This combination is easy to compute and it is given by $\langle n-1 | P_{n,i} | n \rangle$.

In practice, it is very useful to note the following. Due to our choice of reference spinors in (2.3), one can easily show that diagrams with an upper vertex of the form $(++-)$ or with a lower vertex of the form $(--+)$ vanish. For example, assume that the $(n-2)$ -th gluon has positive helicity. Then the first term in figure 1 vanishes for $i = n-3$, i.e., the term with upper vertex $((n-2)^+, \widehat{n-1}^-, \widehat{P}_{n,n-3}^+)$.

We illustrate this procedure in detail with calculation of a six-gluon amplitude below; the same procedure was used in the computation of all the results in sections 3 and 4.

2.2. An Explicit Example

As a first application of our formula, we compute the next-to-MHV six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. Note that we have shifted the labels with respect to the conventions in the previous section. This is done in order to compare more easily to the result to the known formula in the literature [3,4].

Here we choose the reference gluons to be $\widehat{3}$ and $\widehat{4}$. There are three possible configurations of external gluons. Only one helicity configuration for the internal gluon gives a nonzero answer.

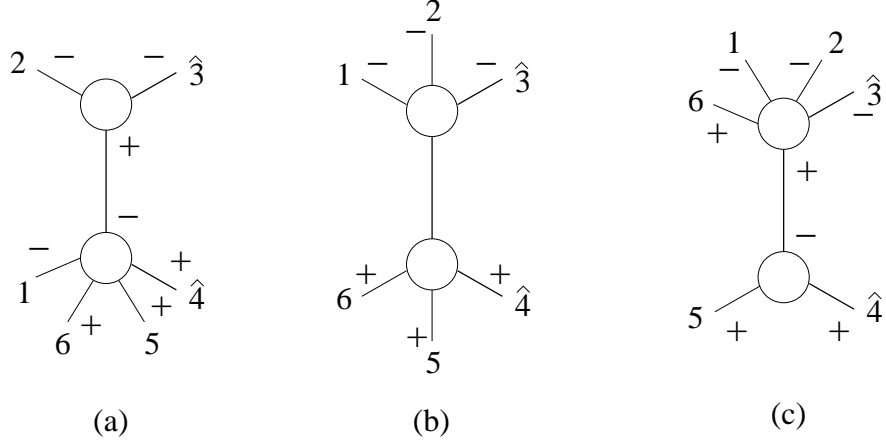


Fig. 2: Configurations contributing to the six-gluon amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. Note that (a) and (c) are related by a flip and a conjugation. (b) vanishes for either helicity configuration of the internal line.

This is shown in fig. 2. Note that for this helicity configuration, the middle graph vanishes. Therefore, we are left with only two graphs to evaluate. Moreover, the two graphs are related by a flip of indices composed with a conjugation. Therefore, only one computation is needed.

Let us compute in detail the contribution coming from the first graph shown in fig. 2(a). The contribution of this term is given by the product of two MHV amplitudes times a propagator,

$$\left(\frac{\langle 2 \hat{3} \rangle^3}{\langle \hat{3} \hat{P} \rangle \langle \hat{P} 2 \rangle} \right) \frac{1}{t_2^{[2]}} \left(\frac{\langle 1 \hat{P} \rangle^3}{\langle \hat{P} \hat{4} \rangle \langle \hat{4} 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle} \right). \quad (2.6)$$

This formula can be simplified by noting that

$$\begin{aligned} \lambda_{\hat{3}} &= \lambda_3, \\ \lambda_{\hat{4}} &= \lambda_4 - \frac{t_2^{[2]}}{\langle 3 2 \rangle [2 4]} \lambda_3, \\ \langle \bullet \hat{P} \rangle &= -\frac{\langle \bullet | 2 + 3 | 4 \rangle}{[\hat{P} 4]}. \end{aligned} \quad (2.7)$$

Using (2.7) it is straightforward to find (2.6)

$$\frac{\langle 1 | 2 + 3 | 4 \rangle^3}{[2 3][3 4] \langle 5 6 \rangle \langle 6 1 \rangle t_2^{[3]} \langle 5 | 3 + 4 | 2 \rangle}. \quad (2.8)$$

Finally, applying the flip that takes $i \rightarrow i + 3$ and a conjugation, i.e., $\langle \rangle \leftrightarrow []$, to (2.8) we find the contribution from configuration (c). Adding both contributions and factoring out a common term, we get

$$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{1}{\langle 5|3+4|2 \rangle} \left(\frac{\langle 1|2+3|4 \rangle^3}{[2\ 3][3\ 4]\langle 5\ 6 \rangle \langle 6\ 1 \rangle t_2^{[3]}} + \frac{\langle 3|4+5|6 \rangle^3}{[6\ 1][1\ 2]\langle 3\ 4 \rangle \langle 4\ 5 \rangle t_3^{[3]}} \right). \quad (2.9)$$

Quite surprisingly, this is the formula found in [18] by taking a collinear limit of a seven-gluon amplitude representation given in [16].

3. Previously Known Amplitudes

In this section we recompute all known tree-level amplitudes of gluons for $n \leq 7$ using our recursion relations. It turns out that all the formulas we get come out naturally in a very compact form. We start with the MHV amplitudes and show that they satisfy the recursion relation. For next-to-MHV amplitudes, we compute all six- [3,4] and seven-gluon amplitudes [5]. Finally, we compute the next-to-next-to-MHV eight gluon amplitude with four adjacent minuses [18].

All the results presented in this section and the next were computed using exactly the same technique as in the example of section 2.2. Here we will not repeat the details but we will indicate explicitly the contribution coming from each term in (2.1). It turns out that in all the cases considered here only one helicity configuration of the internal propagator gives a non-zero contribution. Therefore, in order to specify a given term in (2.1), it is enough to give the gluons in each tree amplitude and the reference gluons denoted by a hat. In the example of section 2.2, we would refer to the contribution from fig. 2(a) as $(2, \widehat{3}|\widehat{4}, 5, 6, 1)$ and from fig. 2(c) as $(6, 1, 2, \widehat{3}|\widehat{4}, 5)$.

3.1. MHV Amplitudes

We show here that the Parke-Taylor formula for MHV amplitudes (1.1) satisfies the recursion relations.

Consider the amplitude $A(1^-, 2^+, \dots, (j-1)^+, j^-, (j+1)^+, \dots, (n-1)^+, n^+)$ and assume that (1.1) is valid for all MHV amplitudes with fewer than n gluons.⁴ Using the

⁴ Of course, we also assume that the conjugate amplitudes for fewer than n gluons are also valid.

recursion relation (2.1), with 1 and 2 chosen as the reference gluons, we find that only one term is non-zero. It is the term given by $(4, 5, \dots, n, \widehat{1} \mid \widehat{2}, 3)$. Its contribution is given by

$$\frac{\langle j \widehat{1} \rangle^4}{\langle 4 \ 5 \rangle \dots \langle n-1 \ n \rangle \langle n \ \widehat{1} \rangle \langle \widehat{1} \ \widehat{P} \rangle \langle \widehat{P} \ 4 \rangle} \times \frac{1}{\langle 2 \ 3 \rangle [2 \ 3]} \times \frac{[\widehat{2} \ 3]^3}{[3 \ \widehat{P}] [\widehat{P} \ \widehat{2}]}, \quad (3.1)$$

where $P = p_2 + p_3$.

After using (2.4) and (2.5) to remove the hats, we find

$$A(1^-, 2^+, \dots, (j-1)^+, j^-, (j+1)^+, \dots, (n-1)^+, n^+) = \frac{\langle 1 \ j \rangle^4}{\langle 4 \ 5 \rangle \langle 5 \ 6 \rangle \dots \langle n-1 \ n \rangle \langle n \ 1 \rangle \langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 4 \rangle}. \quad (3.2)$$

3.2. Six-Gluon Amplitudes

We now compute all next-to-MHV six-gluon amplitudes.

The case with three adjacent minus helicities was presented in detail in the previous section, and the answer appears in (2.9).

The next configuration gives a three-term expression. These three terms are the contributions from $(2, \widehat{3} \mid \widehat{4}, 5, 6, 1)$, $(1, 2, \widehat{3} \mid \widehat{4}, 5, 6)$, and $(6, 1, 2, \widehat{3} \mid \widehat{4}, 5)$ respectively.

$$\begin{aligned} A(1^+, 2^+, 3^-, 4^+, 5^-, 6^-) &= \frac{[2 \ 4]^4 \langle 5 \ 6 \rangle^3}{[2 \ 3][3 \ 4] \langle 6 \ 1 \rangle t_2^{[3]} \langle 1 \mid 2 + 3 \mid 4 \rangle \langle 5 \mid 3 + 4 \mid 2 \rangle} \\ &+ \frac{\langle 3 \mid 1 + 2 \mid 4 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle [4 \ 5] [5 \ 6] t_1^{[3]} \langle 1 \mid 2 + 3 \mid 4 \rangle \langle 3 \mid 1 + 2 \mid 6 \rangle} \\ &+ \frac{[1 \ 2]^3 \langle 3 \ 5 \rangle^4}{[6 \ 1] \langle 3 \ 4 \rangle \langle 4 \ 5 \rangle t_3^{[3]} \langle 5 \mid 3 + 4 \mid 2 \rangle \langle 3 \mid 4 + 5 \mid 6 \rangle}. \end{aligned} \quad (3.3)$$

Similarly, for the final configuration of a next-to-MHV six-gluon amplitude, there are three terms that are the contributions from $(1, \widehat{2} \mid \widehat{3}, 4, 5, 6)$, $(6, 1, \widehat{2} \mid \widehat{3}, 4, 5)$, and $(5, 6, 1, \widehat{2} \mid \widehat{3}, 4)$ respectively.

$$\begin{aligned} A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-) &= \frac{[1 \ 3]^4 \langle 4 \ 6 \rangle^4}{[1 \ 2][2 \ 3] \langle 4 \ 5 \rangle \langle 5 \ 6 \rangle t_1^{[3]} \langle 6 \mid 1 + 2 \mid 3 \rangle \langle 4 \mid 2 + 3 \mid 1 \rangle} \\ &+ \frac{\langle 2 \ 6 \rangle^4 [3 \ 5]^4}{\langle 6 \ 1 \rangle \langle 1 \ 2 \rangle [3 \ 4] [4 \ 5] t_3^{[3]} \langle 6 \mid 4 + 5 \mid 3 \rangle \langle 2 \mid 3 + 4 \mid 5 \rangle} \\ &+ \frac{[1 \ 5]^4 \langle 2 \ 4 \rangle^4}{\langle 2 \ 3 \rangle \langle 3 \ 4 \rangle [5 \ 6] [6 \ 1] t_2^{[3]} \langle 4 \mid 2 + 3 \mid 1 \rangle \langle 2 \mid 3 + 4 \mid 5 \rangle}. \end{aligned} \quad (3.4)$$

These two expressions have been checked against the known results [3]. It is amazing to notice that for $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-)$, the second two terms can be obtained from the first by shifting all indices by $i \rightarrow i + 2$ and $i \rightarrow i + 4$.

3.3. Seven-Gluon Amplitudes

Now we use our recursion relation to calculate the tree level next-to-MHV amplitude of seven gluons and compare with results given in [16]. We follow the conventions of that paper to write the four independent helicity configurations.

For configuration A: $(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$, there are only two nonzero contributions, namely from $(2, \widehat{3}|\widehat{4}, 5, 6, 7, 1)$ and $(6, 7, 1, 2, \widehat{3}|\widehat{4}, 5)$. The first involves only MHV amplitudes, so it is just one term. The second involves the next-to-MHV six-gluon amplitude with two terms. We write these three terms in order here:

$$\begin{aligned}
A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) = & \\
& \frac{\langle 1|2+3|4\rangle^3}{t_2^{[3]}\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle[2\ 3][3\ 4]\langle 5|4+3|2\rangle} \\
& - \frac{1}{\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 6|7+1|2\rangle} \left(\frac{\langle 3|(4+5)(6+7)|1\rangle^3}{t_3^{[3]}t_6^{[3]}\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 5|4+3|2\rangle} + \frac{\langle 3|2+1|7\rangle^3}{t_7^{[3]}\langle 6\ 5\rangle[7\ 1][1\ 2]} \right). \tag{3.5}
\end{aligned}$$

Term by term, this expression is equal to $c_B + c_{347}|_{\text{flip}} + c_{347}$ from [16], which is exactly the compact formula given there. For configuration B: $(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$, there are three nonzero contributions. We write the formula in the following order: the single term from $(3, \widehat{4}|\widehat{5}, 6, 7, 1, 2)$, the single term from $(2, 3, \widehat{4}|\widehat{5}, 6, 7, 1)$, and the three terms from $(7, 1, 2, 3, \widehat{4}|\widehat{5}, 6)$.

$$\begin{aligned}
A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+) = & \\
& \frac{\langle 1\ 2\rangle^3[3\ 5]^4}{t_3^{[3]}[3\ 4][4\ 5]\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5\rangle\langle 6|4+5|3\rangle} \\
& + \frac{\langle 2\ 4\rangle^4\langle 1|7+6|5\rangle^3}{t_2^{[3]}t_6^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5\rangle\langle 6|(7+1)(2+3)|4\rangle} \\
& + \frac{\langle 1\ 2\rangle^3\langle 4|5+6|3\rangle^4}{t_4^{[3]}t_7^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle\langle 6|4+5|3\rangle\langle 7|1+2|3\rangle\langle 4|(5+6)(7+1)|2\rangle} \\
& + \frac{\langle 4|1+2|3\rangle^4}{t_1^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 4|3+2|1\rangle\langle 7|1+2|3\rangle} \\
& + \frac{\langle 2\ 4\rangle^4\langle 4|5+6|7\rangle^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle[7\ 1]\langle 4|3+2|1\rangle\langle 4|(5+6)(7+1)|2\rangle\langle 6|(7+1)(2+3)|4\rangle}. \tag{3.6}
\end{aligned}$$

Term by term, this expression is equal to $c_{145} + c_A + c_E + c_{236} + c_{136}$ from [16]. This is not the exact compact formula given in that paper, but it is possible to derive from the

relations given there that this is the correct tree amplitude. It would be interesting to check if a different choice of reference gluons reproduces the compact form of [16], term by term.

For configuration C: $(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+)$, there are four nonzero contributions. We write the formula in the following order: the single term from $(1, \widehat{2}\widehat{3}, 4, 5, 6, 7)$, the single term from $(7, 1, \widehat{2}\widehat{3}, 4, 5, 6)$, the single term from $(6, 7, 1, \widehat{2}\widehat{3}, 4, 5)$, and the three terms from $(5, 6, 7, 1, \widehat{2}\widehat{3}, 4)$.

$$\begin{aligned}
A(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+) = & \\
& \frac{\langle 5|1+2|3\rangle^4}{t_1^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7|1+2|3\rangle\langle 4|3+2|1\rangle} \\
& + \frac{\langle 1\ 2\rangle^3\langle 5|4+6|3\rangle^4}{t_4^{[3]}t_7^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle\langle 6|5+4|3\rangle\langle 7|1+2|3\rangle\langle 4|(5+6)(7+1)|2\rangle} \\
& + \frac{\langle 1\ 2\rangle^3[3\ 4]^3}{t_3^{[3]}[4\ 5]\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5\rangle\langle 6|4+5|3\rangle} \\
& + \frac{\langle 1\ 2\rangle^3\langle 2|3+4|6\rangle^4}{\langle 7\ 1\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6]\langle 2|1+7|6\rangle\langle 2|3+4|5\rangle\langle 2|(3+4)(5+6)|7\rangle\langle 4|(5+6)(7+1)|2\rangle} \\
& + \frac{\langle 2|(3+4)(7+6)|5\rangle^4}{t_2^{[3]}t_5^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 5|6+7|1\rangle\langle 4|2+3|1\rangle\langle 2|(3+4)(5+6)|7\rangle} \\
& + \frac{\langle 2\ 5\rangle^4[6\ 7]^3}{t_6^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle[7\ 1]\langle 5|6+7|1\rangle\langle 2|7+1|6\rangle}.
\end{aligned} \tag{3.7}$$

Term by term, this expression is equal to $c_{236} + c_A|_{\text{flip}} + c_{367}|_{\text{flip}} + c_{357}|_{\text{flip}} + c_C|_{\text{flip}} + c_{147}$ from [16]. It is possible to derive from the formulas in that paper that this is the correct tree amplitude.

For configuration D: $(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+)$, there are again four nonzero contributions. We write the formula in the following order: the single term from $(2, \widehat{3}\widehat{4}, 5, 6, 7, 1)$, the single term from $(1, 2, \widehat{3}\widehat{4}, 5, 6, 7)$, the single term from $(7, 1, 2, \widehat{3}\widehat{4}, 5, 6)$, and the three

terms from $(6, 7, 1, 2, \widehat{3}\widehat{4}, 5)$.

$$\begin{aligned}
A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = & \frac{\langle 1\ 5 \rangle^4 [2\ 4]^4}{t_2^{[3]} [2\ 3] [3\ 4] \langle 5\ 6 \rangle \langle 6\ 7 \rangle \langle 7\ 1 \rangle \langle 1|2+3|4 \rangle \langle 5|3+4|2 \rangle} \\
& - \frac{\langle 1\ 3 \rangle^4 \langle 5|6+7|4 \rangle^4}{t_1^{[3]} t_5^{[3]} \langle 1\ 2 \rangle \langle 2\ 3 \rangle \langle 5\ 6 \rangle \langle 6\ 7 \rangle \langle 1|2+3|4 \rangle \langle 7|5+6|4 \rangle \langle 5|(6+7)(1+2)|3 \rangle} \\
& + \frac{\langle 1\ 3 \rangle^4 [4\ 6]^4}{t_4^{[3]} \langle 7\ 1 \rangle \langle 1\ 2 \rangle \langle 2\ 3 \rangle [4\ 5] [5\ 6] \langle 3|4+5|6 \rangle \langle 7|5+6|4 \rangle} \\
& - \frac{\langle 3\ 5 \rangle^4 [2\ 7]^4}{t_7^{[3]} \langle 3\ 4 \rangle \langle 4\ 5 \rangle \langle 5\ 6 \rangle [7\ 1] [1\ 2] \langle 6|7+1|2 \rangle \langle 3|2+1|7 \rangle} \\
& + \frac{\langle 3\ 5 \rangle^4 \langle 1|6+7|2 \rangle^4}{t_3^{[3]} t_6^{[3]} \langle 6\ 7 \rangle \langle 7\ 1 \rangle \langle 3\ 4 \rangle \langle 4\ 5 \rangle \langle 6|7+1|2 \rangle \langle 5|3+4|2 \rangle \langle 3|(4+5)(6+7)|1 \rangle} \\
& - \frac{\langle 1\ 3 \rangle^4 \langle 3\ 5 \rangle^4 [6\ 7]^3}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \langle 3\ 4 \rangle \langle 4\ 5 \rangle \langle 3|4+5|6 \rangle \langle 3|2+1|7 \rangle \langle 3|(4+5)(6+7)|1 \rangle \langle 5|(6+7)(1+2)|3 \rangle}.
\end{aligned} \tag{3.8}$$

Term by term, this expression is equal to $c_{347} + c_B + c_{256} + c_{256|\text{flip}} + c_{B|\text{flip}} + c_{257}$ from [16]. This is exactly the same compact formula for the tree amplitude given in that paper.

3.4. Eight-Gluon Amplitude

A very compact formula for the next-to-next-to-MHV amplitude $A(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$ was computed very recently in [18].

Using our formula (2.1) it is easy to see that there are only two terms contributing to the amplitude. They are $(3, \widehat{4}\widehat{5}, 6, 7, 8, 1, 2)$ and $(7, 8, 1, 2, 3, \widehat{4}\widehat{5}, 6)$. Let us first consider the contribution $(3, \widehat{4}\widehat{5}, 6, 7, 8, 1, 2)$. Using the seven-gluon amplitude (3.5), we find immediately that

$$\begin{aligned}
I_a = & \frac{\langle 1|K_2^{[3]}|5 \rangle^3}{t_2^{[4]} [2\ 3] [3\ 4] [4\ 5] \langle 6\ 7 \rangle \langle 7\ 8 \rangle \langle 8\ 1 \rangle \langle 6|K_3^{[3]}|2 \rangle} \\
& - \frac{\langle 1|K_7^{[2]} K_3^{[4]} K_3^{[2]}|5 \rangle^3}{t_7^{[3]} t_3^{[4]} t_3^{[3]} [3\ 4] [4\ 5] \langle 7\ 8 \rangle \langle 8\ 1 \rangle \langle 6|K_4^{[2]}|3 \rangle \langle 7|K_8^{[2]}|2 \rangle \langle 6|K_3^{[3]}|2 \rangle} \\
& - \frac{[8|K_1^{[2]} K_3^{[2]}|5 \rangle^3}{\langle 7\ 6 \rangle [8\ 1] [1\ 2] [3\ 4] [4\ 5] t_3^{[3]} t_8^{[3]} \langle 7|K_8^{[2]}|2 \rangle \langle 6|K_4^{[2]}|3 \rangle},
\end{aligned} \tag{3.9}$$

where the order of these three terms is the same as in (3.5). Our convention here is that $K_i^{[r]} \equiv (p_i + p_{i+1} + \dots + p_{i+r-1})$. The contribution from $(7, 8, 1, 2, 3, \widehat{4}\widehat{5}, 6)$ is just the flip of I_a by relabeling $i \rightarrow (9-i)$ and exchanging $\langle \rangle$ and $[\]$.

Let us compare our result with the one given in equation (1) of [18]. It is easy to see that the first two terms match, while the last term in (3.9) is related by the above flip to the term written explicitly in [18].

4. Result for $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-)$

In this section we present the NNMHV eight-gluon amplitude with alternating helicities. There are five different configurations of gluons we have to consider. Here again, to save space, we use the notation $K_i^{[r]} \equiv (p_i + p_{i+1} + \dots + p_{i+r-1})$.

First we find the contribution of $(1, \widehat{2}\widehat{3}, 4, 5, 6, 7, 8)$. It is

$$\begin{aligned}
I_1 = & \frac{[1\ 3]^4 \langle 4\ 6 \rangle^4 \langle 6\ 8 \rangle^4}{[1\ 2][2\ 3]\langle 4\ 5 \rangle \langle 5\ 6 \rangle \langle 6\ 7 \rangle \langle 7\ 8 \rangle \langle 6|K_7^{[2]}|1\rangle \langle 6|K_4^{[2]}|3\rangle \langle 6|K_7^{[2]}K_1^{[3]}|4\rangle \langle 8|K_1^{[3]}K_4^{[2]}|6\rangle} \\
& + \frac{[1\ 3]^4 [5\ 7]^4 \langle 4\ 8 \rangle^4}{[1\ 2][2\ 3][5\ 6][6\ 7]t_1^{[3]}t_5^{[3]}\langle 4|K_2^{[2]}|1\rangle \langle 4|K_5^{[2]}|7\rangle \langle 8|K_1^{[2]}|3\rangle \langle 8|K_6^{[2]}|5\rangle} \\
& - \frac{[1\ 3]^4 \langle 4\ 6 \rangle^4 \langle 8|K_1^{[3]}|7\rangle^4}{[1\ 2][2\ 3]\langle 4\ 5 \rangle \langle 5\ 6 \rangle t_1^{[3]}t_4^{[3]}t_8^{[4]}\langle 4|K_5^{[2]}|7\rangle \langle 8|K_1^{[2]}|3\rangle [1|K_2^{[2]}K_4^{[3]}|7\rangle \langle 8|K_1^{[3]}K_4^{[2]}|6\rangle} \\
& + \frac{[1\ 3]^4 [7\ 1]^4 \langle 4\ 6 \rangle^4}{[1\ 2][2\ 3]\langle 4\ 5 \rangle \langle 5\ 6 \rangle [7\ 8][8\ 1]\langle 4|K_2^{[2]}|1\rangle \langle 6|K_7^{[2]}|1\rangle [1|K_2^{[2]}K_4^{[3]}|7\rangle [3|K_4^{[3]}K_7^{[2]}|1\rangle} \\
& - \frac{[1\ 3]^4 \langle 6\ 8 \rangle^4 \langle 4|K_1^{[3]}|5\rangle^4}{[1\ 2][2\ 3]\langle 6\ 7 \rangle \langle 7\ 8 \rangle t_1^{[3]}t_6^{[3]}t_5^{[4]}\langle 4|K_2^{[2]}|1\rangle \langle 8|K_6^{[2]}|5\rangle [5|K_6^{[3]}K_1^{[2]}|3\rangle \langle 6|K_7^{[2]}K_1^{[3]}|4\rangle} \\
& + \frac{[1\ 3]^4 [3\ 5]^4 \langle 6\ 8 \rangle^4}{[1\ 2][2\ 3][3\ 4][4\ 5]\langle 6\ 7 \rangle \langle 7\ 8 \rangle \langle 6|K_4^{[2]}|3\rangle \langle 8|K_1^{[2]}|3\rangle [3|K_4^{[2]}K_6^{[3]}|1\rangle [5|K_6^{[3]}K_1^{[2]}|3\rangle}.
\end{aligned} \tag{4.1}$$

From this one we can get the contribution of $(5, 6, 7, 8, 1, \widehat{2}\widehat{3}, 4)$ by a conjugate flip operation that exchanges indices $2 \leftrightarrow 3$, $1 \leftrightarrow 4$, $8 \leftrightarrow 5$, $7 \leftrightarrow 6$ and $\langle \rangle \leftrightarrow []$.

Second, we get that the contribution of $(8, 1, \widehat{2}\widehat{3}, 4, 5, 6, 7)$ is

$$\begin{aligned}
I_2 = & -\frac{\langle 8\ 2 \rangle^4}{\langle 8\ 1 \rangle \langle 1\ 2 \rangle \langle 8|K_8^{[3]}|3\rangle t_8^{[3]}} \left(\frac{[3\ 5]^4 \langle 6|K_8^{[3]}|3\rangle^4}{[3\ 4][4\ 5]\langle 6\ 7 \rangle \langle 7|K_8^{[3]}|3\rangle \langle 6|K_4^{[2]}|3\rangle \langle 2|K_8^{[3]}K_6^{[2]}K_4^{[2]}|3\rangle [5|K_6^{[2]}K_8^{[3]}|3\rangle} \right. \\
& + \frac{[5\ 7]^4 \langle 4|K_8^{[3]}|3\rangle^4}{[5\ 6][6\ 7]t_8^{[4]}t_5^{[3]}\langle 4|K_5^{[3]}K_8^{[3]}|2\rangle [5|K_5^{[3]}K_8^{[3]}|3\rangle \langle 4|K_5^{[3]}|7\rangle} \\
& \left. - \frac{[3\ 7]^4 \langle 4\ 6 \rangle^4}{\langle 4\ 5 \rangle \langle 5\ 6 \rangle t_4^{[3]}\langle 2|K_8^{[3]}|7\rangle \langle 6|K_4^{[3]}|3\rangle \langle 4|K_4^{[3]}|7\rangle} \right).
\end{aligned} \tag{4.2}$$

From this expression, by the same conjugate flip, we can get the contribution from $(6, 7, 8, 1, \widehat{2}|\widehat{3}, 4, 5)$.

Finally, from $(7, 8, 1, \widehat{2}|\widehat{3}, 4, 5, 6)$ we get

$$I_3 = -\frac{\langle 8\ 2 \rangle^4}{\langle 7\ 8 \rangle \langle 8\ 1 \rangle \langle 1\ 2 \rangle \langle 7|K_7^{[4]}|3\rangle t_7^{[4]}} \frac{1}{[3\ 4][4\ 5][5\ 6]\langle 2|K_3^{[4]}|6\rangle} \\ - \frac{[7\ 1]^4 \langle 2|K_7^{[4]}|3\rangle^2}{[7\ 8][8\ 1]t_7^{[3]}[3|K_3^{[4]}K_7^{[2]}|1]\langle 2|K_7^{[4]}|7\rangle t_7^{[4]}} \frac{1}{\langle 4\ 5 \rangle \langle 5\ 6 \rangle t_4^{[3]} \langle 4|K_5^{[2]}K_7^{[4]}|2\rangle \langle 6|K_3^{[4]}|3\rangle}.$$
(4.3)

Notice that there are two terms corresponding to two different helicity assignments on the propagator.

The final result is

$$A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-) = [I_1 + I_1^{\text{conj.flip}}] + [I_2 + I_2^{\text{conj.flip}}] + I_3. \quad (4.4)$$

4.1. Symmetric Form

The amplitude $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-)$ has a high degree of symmetry. Our recursion procedure breaks almost all the symmetry by choosing two reference gluons. Here we show that from the formula above, we can in fact deduce a fully symmetric expression.

The full symmetry group is the dihedral group of order 16, with the following two generators acting on the gluon indices:

$$g : i \rightarrow i + 1 \text{ with conjugation}, \quad r : i \rightarrow -i \pmod{8}. \quad (4.5)$$

These satisfy the relations $g^8 = 1, r^2 = 1$ and $rgrg = 1$. Many of the terms of (4.1), (4.2), and (4.3) are related by these symmetries. Thus it is possible to represent those equations, term by term, as follows:

$$I_1 = T + U + V + g^3 T + g^5 V + g^5 T \\ I_2 = W + g^4 V + g^3 U \\ I_3 = X + g^3 V.$$
(4.6)

The five independent terms we have defined have the following symmetry:

$$T = g^4 r T, \quad U = g^4 U = r U, \quad V = g^7 r V, \quad X = g^5 r X. \quad (4.7)$$

The flip symmetry used in (4.4) is $g^5 r$ in this notation. After performing this action on I_1 and I_2 to get the amplitude (4.4) and using the relations in (4.7), we can reorder the terms to get the expression

$$\begin{aligned} A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-) = & (T + gT + g^3T + g^4T + g^5T + g^6T) \\ & + (U + gU + g^2U + g^3U) + (V + gV + g^2V + g^3V + g^4V + g^5V + g^6V) \\ & + (W + g^5rW) + X. \end{aligned} \tag{4.8}$$

We have checked that the last three terms, $(W + g^5rW) + X$, are equal to the sum $g^2T + g^7T + g^7V$. Making this substitution in (4.8) gives an expression with all the required symmetry manifest. The amplitude is the sum of the orbits of the terms T, U and V .

5. Future Directions

In this section we present some of the future directions that are natural to explore given the success of the recursion relation (2.1).

First we give an outline of a possible proof of the recursion relation (2.1). Then we show how one can use the recursion relation several times to write any amplitude as the sum of terms computed from only trivalent vertices with helicities $(+ + -)$ and $(- - +)$. These new diagrams hint at a connection to a string theory whose target space is the Quadric. We also comment on a possible connection to MHV diagrams. We give a set of amplitudes that are closed under the recursion relations and suggest that one can hope to solve it explicitly. Finally, we comment on a possible extension of the recursion relations that involves reference gluons of the same helicity.

5.1. Outline of Possible Proof

As stated in the introduction, our conjectured recursion relations are based on recent results for calculating one-loop $\mathcal{N} = 4$ amplitudes [19] and some older results describing their infrared behavior [15]. In particular, the infrared behavior is given by

$$A_{n:1}^{1\text{-loop}}|_{\text{IR}} = \left[-\frac{1}{\epsilon^2} \sum_{j=1}^n \left(-t_j^{[2]} \right)^{-\epsilon} \right] A_{n:0}^{\text{tree}}. \tag{5.1}$$

Our ability to extract from (5.1) a recursion relation from tree amplitudes is due to the recent discovery in [19] that any box integral coefficient can be expressed simply as a product of four tree amplitudes by the formula

$$\hat{a}_\alpha = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S}, J} n_J A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}. \quad (5.2)$$

The four amplitudes in the product correspond to the four corners of the box. This formula was derived from considering the quadruple cut in the theory with signature $(++--)$.

From the infrared behavior (5.1) it is possible to derive the relation (1.2) [18].

$$A_n^{\text{tree}} = \frac{1}{2} \sum_{i=1}^{n-3} B_{1,i+1,n-1,n}. \quad (5.3)$$

Note that each box integral whose coefficient appears in (5.3) has (at least) two adjacent trivalent vertices.

The crucial point of our conjecture is that the factor of $\frac{1}{2}$ in (5.3) has a deep meaning. It suggests that the sum splits naturally into two groups. We call them the \mathcal{A} and the \mathcal{B} group. Schematically, one can formally obtain

$$A_n = \frac{1}{2} \sum_{i=1}^{n-3} B_{1,i+1,n-1,n}^{\mathcal{A}} + \frac{1}{2} \sum_{i=1}^{n-3} B_{1,i+1,n-1,n}^{\mathcal{B}}. \quad (5.4)$$

The two groups are defined and distinguished by the helicity assignments at the adjacent trivalent vertices. In (5.2), \mathcal{S} is the set of solutions of momenta in the cut propagators. There are two solutions (given explicitly in [19]), and each determines the type of helicity assignment $(++- \text{ or } --+)$ allowed at each trivalent vertex. The \mathcal{A} group is defined as the one for which only gluons can circulate in the loop. For the \mathcal{B} group the whole $\mathcal{N} = 4$ multiplet is allowed (but not necessary realized). (See the appendix for further details.) This natural separation motivated us to propose the \mathcal{A} (or equivalently the \mathcal{B}) conjecture: each set of terms in (5.4) is enough to reproduce the whole amplitude, i.e.,

$$A_n = \sum_{i=1}^{n-3} B_{1,i+1,n-1,n}^{\mathcal{A}} = \sum_{i=1}^{n-3} B_{1,i+1,n-1,n}^{\mathcal{B}}. \quad (5.5)$$

The \mathcal{A} part of this conjecture is the recursion relations proposed in section 2 and used to obtain all the amplitudes in sections 3 and 4. It is amusing to realize that another (perhaps not very useful) recursion relation can be written down from the \mathcal{B} terms. The

tree-amplitudes there would contain fermions and scalars as external legs. Adding all these terms would give the amplitude with only gluons.

To complete this proof, it would be necessary to check that the \mathcal{A} and \mathcal{B} sums are equal. It is natural to expect that this is a consequence of applying Ward identities to the former to get the latter. It would be interesting to pursue this in the future.

It should be possible to derive similar recursion relations for tree amplitudes with external fermions by applying supersymmetry transformations to the recursion relations proposed here.

Since our recursion relation involves tree-level amplitudes, it should be oblivious to $\mathcal{N} = 4$ supersymmetry. Therefore we believe that it is most naturally given in terms of the \mathcal{A} group.

Equation (5.2) gives the coefficients $B_{1,i+1,n-1,n}^{\mathcal{A}}$ in terms of a product of two three-particle tree amplitudes and two possibly larger tree amplitudes.

The two three-gluon amplitudes can be explicitly reduced to produce the appropriate conversion factor between scalar box integrals and scalar box functions.

For details of this derivation, see the appendix.

5.2. Trivalent-Vertex Representation

The recursion relation (2.1) gives a given amplitude in terms of amplitudes with fewer gluons,

$$A_n(1, 2, \dots, (n-1)^-, n^+) = \sum_{i=1}^{n-3} \sum_{h=+,-} \left(A_{i+2}(\widehat{n}, 1, 2, \dots, i, -\widehat{P}_{n,i}^h) \frac{1}{P_{n,i}^2} A_{n-i}(+\widehat{P}_{n,i}^{-h}, i+1, \dots, n-2, \widehat{n-1}) \right). \quad (5.6)$$

As discussed in section 2, each amplitude on the right hand side of (5.6) is on-shell and momentum conservation is valid. Therefore, we can apply the recursion relation again to all of them to get lower amplitudes. This process can be repeated any number of times until all amplitudes entering in the expression for A_n are reduced to three-gluon amplitudes.

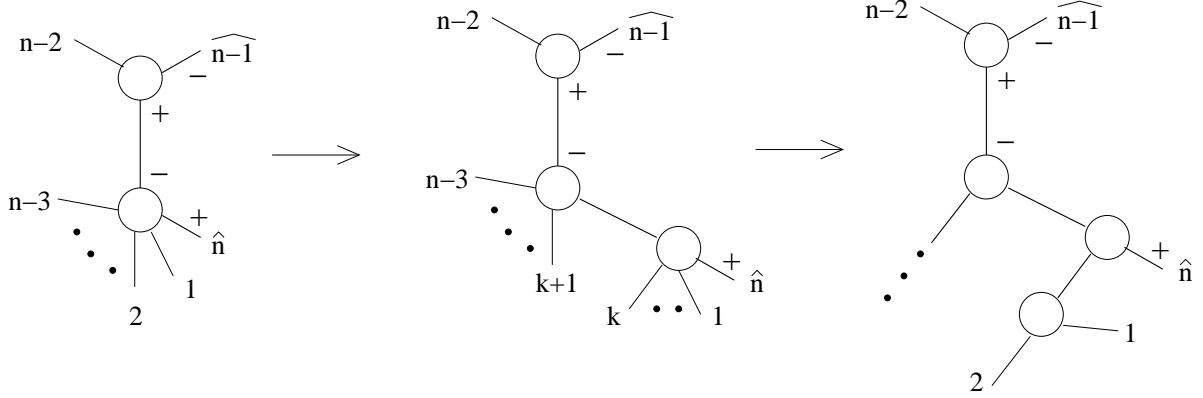


Fig. 3: Schematic representation of the process to reduce all tree-level amplitudes to trees with only $(+ + -)$ and $(- - +)$ vertices.

This process is illustrated in fig. 3.

It is important to note that this decomposition is only possible thanks to the fact that all amplitudes are on-shell and intermediate momenta take values in signature $(- - ++)$ where three-gluon amplitudes on-shell do not vanish.

This decomposition hints at the fascinating possibility that there might be an effective Lagrangian describing classical gauge theory in terms of a scalar field with only cubic interactions.

Another tempting conjecture is that the diagrams obtained using only the amplitudes $(+ + -)$ and $(- - +)$ are the natural outcome of a string theory whose target space is the super Calabi-Yau manifold known as the Quadric. Recall that the original twistor string theory proposed in [7] is a topological B model with target space the super Calabi-Yau manifold $\mathbb{CP}^{3|4}$. Also in [7], an alternative to this target space was suggested. Take two copies of $\mathbb{CP}^{3|3}$, one with homogeneous coordinates Z^I, ψ^A and the other with W^I, χ^A with $I = 1, \dots, 4$, $A = 1, \dots, 3$. Then the Quadric is the zero set given by

$$\sum_{I=1}^4 Z^I W_I + \sum_{A=1}^3 \psi^A \chi_A = 0 \quad (5.7)$$

in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. Note that the original topological B model with target space $\mathbb{CP}^{3|4}$ is not manifestly parity invariant and has to be enriched with D-instantons in order to reproduce tree-level amplitudes. In contrast, the string theory on the Quadric is manifestly parity invariant and requires no D-instantons [7].

It is also natural to expect that a connection to the MHV diagram (CSW) construction of tree amplitudes can be made from the trivalent representation of the amplitudes. The rough idea is based on two conjectures once the representation in terms of trivalent vertices is given:

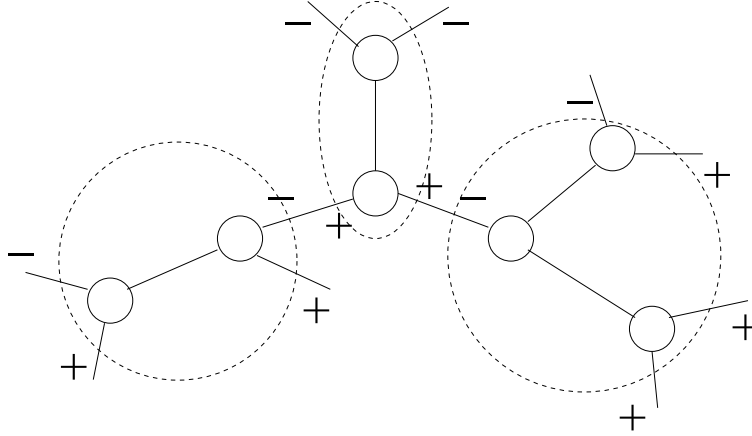


Fig. 4: Possible connection between the trivalent representation of tree amplitudes and MHV diagrams. Note that the helicities of the gluons connected to each circle have MHV structure.

One is that it might always be possible to circle connected set of vertices such that the legs coming out of the circle have MHV-like helicities (see fig. 4). The second is that all the vertices inside one such circle will simplify to produce an MHV amplitude. That this might be true is suggested by the computation in section 3.1, along with the fact that each leg coming out of the circle is on-shell and the sum of the momenta is zero.

5.3. Closed Set of Amplitudes

The recursion relations (2.1) give any amplitude in terms of amplitude with fewer gluons but with generic helicity. However, it turns out that there is a set of amplitudes that closes under the recursion procedure. In other words, a given amplitude in the set is determined only by amplitudes in the set.

The set we found is given by amplitudes of the form

$$A_{p,q} = A(1^-, 2^-, \dots, p^-, (p+1)^+, \dots, (p+q)^+) \quad (5.8)$$

for any integers $p \geq 1$ and $q \geq 1$.

Let us study the term contributing to the recursion formula (2.1) when the reference gluons are taken to be p^- and $(p+1)^+$. It is not difficult to check that only two terms are nonzero. They are given by

$$\begin{aligned} &((p-1), \widehat{p} \mid \widehat{p+1}, \dots, (p+q), 1, \dots, (p-2)), \\ &((p+3), \dots, (p+q), 1, \dots, \widehat{p} \mid \widehat{p+1}, (p+2)). \end{aligned} \quad (5.9)$$

The first term is given by $A_{p-1,q}$ times $A_{2,1}$. The second term is given by $A_{p,q-1}$ times $A_{1,2}$.

This proves that the set of amplitudes (5.8) closes under (2.1). It would be very interesting to find an explicit solution to these equations.

One simple observation is that the number of terms in $A_{p,q}$, which we denote by $N_{p,q}$, satisfy the following recursion relation: $N_{p,q} = N_{p-1,q} + N_{p,q-1}$ with boundary conditions $N_{2,q} = 1, \forall q \geq 1$ and $N_{p,2} = 1, \forall p \geq 1$. Thus we recognize the number of terms as a binomial coefficient: $N_{p,q} = (p+q-4)!/((p-2)!(q-2)!)$.

A special case of this closed set is the next-to-MHV amplitudes in the helicity configuration $(- - - + + \cdots +)$. These amplitudes have been written down in [6,9,21]. It would be interesting to check that our recursion relations reproduce these existing results.

5.4. *New Recursion Relations and Linear Trees*

Finally, we want to mention another interesting direction. Recall that the recursion relations (2.1) use as reference vectors two gluons of opposite helicity. One natural question to ask is whether the same formula is valid for reference gluons of the same helicity. We have evidence that this is indeed the case, we have computed all next-to-MHV six-gluon amplitudes (except, of course, $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-)$) and found perfect agreement.

In section 5.2, we described how to iterate (2.1) in order to write any tree amplitude in terms of trivalent vertices. The final answer is given in terms of general trees with trivalent vertices (see fig. 3). But now we can do better. We can use any two gluons as reference vectors, regardless of their helicity, and make the following sequence of choices, $((n-1), n), (n, 1), (1, 2)$, and so on. By doing this we produce only linear trees! It would be interesting to prove that this simple picture reproduces all known amplitudes.

Acknowledgments

It is a pleasure to thank M. Spradlin, P. Svrček, A. Volovich and E. Witten for useful discussions. R. B. and B. F. were supported by NSF grant PHY-0070928. F. C. was supported in part by the Martin A. and Helen Chooljian Membership at the Institute for Advanced Study and by DOE grant DE-FG02-90ER40542.

Appendix A. From box integrals to tree amplitudes

Here we show how the coefficient $B_{1,i+1,n-1,n}$ may be written as the product of two tree amplitudes times a propagator. We follow the notation and conventions of [19], which includes the necessary background material. In particular, we will use the shorter notation $d_{r,i}$ to represent these coefficients, where in contrast to our previous notation, r may now take values all the way from 1 to $n-3$. In this appendix, we fix the massless legs to be $(i-2)$ and $(i-1)$.

The plan of this appendix is the following. First we solve the loop momenta ℓ_i , $i = 1, 2, 3, 4$ explicitly. Then we find the box coefficients for the configuration where both massless legs $(i-2)$ and $(i-1)$ have equal (positive) helicities. Finally we find the box coefficients for the configuration $(i-2)^- (i-1)^+$ in terms of two tree amplitudes times a propagator.

A.1. Solving for the loop momentum

Here we solve for the loop momentum explicitly. Let us start from the configuration (a), where the K_3 vertex has helicity distribution $(++-)$ while the K_4 vertex has helicity distribution $(--+)$. We must then have the following relationships (for definitions of ℓ_i, K_i , see Figure 5):

$$\begin{aligned}\lambda_{\ell_4} &= \alpha\lambda_{i-2}, & \lambda_{\ell_3} &= \beta\lambda_{i-2}, \\ \tilde{\lambda}_{\ell_4} &= \gamma\tilde{\lambda}_{i-1}, & \tilde{\lambda}_{\ell_1} &= \rho\tilde{\lambda}_{i-1}, \\ \beta\tilde{\lambda}_{\ell_3} &= \tilde{\lambda}_{i-2} + \alpha\gamma\tilde{\lambda}_{i-1}, & \rho\lambda_{\ell_1} &= \alpha\gamma\lambda_{i-2} - \lambda_{i-1}.\end{aligned}\tag{A.1}$$

Using $\ell_2^2 = 0$, we find that

$$\alpha\gamma = \frac{K_{23}^2}{\langle i-2|K_2|i-1\rangle} = \frac{K_{14}^2}{-\langle i-2|K_1|i-1\rangle}.\tag{A.2}$$

To make the calculation easier, we can use scaling freedom to fix $\alpha = \beta = \rho = 1$, so that we have solved the loop momenta as the following:

$$\begin{aligned}\lambda_{\ell_4} &= \lambda_{i-2}, & \tilde{\lambda}_{\ell_4} &= \gamma\tilde{\lambda}_{i-1}, & \ell_4 &= \gamma\lambda_{i-2}\tilde{\lambda}_{i-1}, \\ \lambda_{\ell_3} &= \lambda_{i-2}, & \tilde{\lambda}_{\ell_3} &= \tilde{\lambda}_{i-2} + \gamma\tilde{\lambda}_{i-1}, & \ell_3 &= p_{i-2} + \ell_4, \\ \lambda_{\ell_1} &= \gamma\lambda_{i-2} - \lambda_{i-1}, & \tilde{\lambda}_{\ell_1} &= \tilde{\lambda}_{i-1}, & \ell_1 &= \ell_4 - p_{i-1}, \\ \gamma &= \frac{K_{23}^2}{\langle i-2|K_2|i-1\rangle} = \frac{K_{14}^2}{-\langle i-2|K_1|i-1\rangle}.\end{aligned}\tag{A.3}$$

In particular, we find that

$$\ell_2 = \ell_4 - K_{14} = - \left(K_{14} - \frac{K_{14}^2}{2q \cdot K_{14}} q \right), \quad (\text{A.4})$$

where q is a new momentum defined by $q_{\alpha\dot{\alpha}} = (\lambda_{i-2})_{\alpha}(\tilde{\lambda}_{i-1})_{\dot{\alpha}}$. The meaning inside the parentheses of (A.4) is simply the projection of the massive momentum K_{14} along the direction defined by our two reference spinors. If we call the $\ell_4 = \frac{K_{14}^2}{2q \cdot K_{14}} q$ the transformation of momentum, we see that from (A.3) ℓ_1 is the momentum $-p_{i-1}$ transformed by ℓ_4 . Similarly, ℓ_3 is a transformed version of p_{i-2} .

Now we move to the configuration (b) where the K_3 vertex has helicity distribution $(- - +)$ while the K_4 vertex has helicity distribution $(+ + -)$. Using a similar method, we get the following result:

$$\begin{aligned} \lambda_{\ell_4} &= \lambda_{i-1}, & \tilde{\lambda}_{\ell_4} &= \alpha \tilde{\lambda}_{i-2}, & \ell_4 &= \alpha \lambda_{i-1} \tilde{\lambda}_{i-2}, \\ \lambda_{\ell_3} &= \lambda_{i-2} + \alpha \lambda_{i-1}, & \tilde{\lambda}_{\ell_3} &= \tilde{\lambda}_{i-2}, & \ell_3 &= p_{i-2} + \ell_4, \\ \lambda_{\ell_1} &= \lambda_{i-1}, & \tilde{\lambda}_{\ell_1} &= \alpha \tilde{\lambda}_{i-2} - \tilde{\lambda}_{i-1}, & \ell_1 &= \ell_4 - p_{i-1}, \\ \alpha &= \frac{K_{23}^2}{\langle i-1 | K_2 | i-2 \rangle} = - \frac{K_{14}^2}{\langle i-1 | K_1 | i-2 \rangle}, \end{aligned} \quad (\text{A.5})$$

and

$$\ell_2 = \ell_4 - K_{14} = - \left(K_{14} - \frac{K_{14}^2}{2\tilde{q} \cdot K_{14}} \tilde{q} \right), \quad (\text{A.6})$$

where \tilde{q} is a new momentum defined by $\tilde{q}_{\alpha\dot{\alpha}} = (\lambda_{i-1})_{\alpha}(\tilde{\lambda}_{i-2})_{\dot{\alpha}}$ so that ℓ_2 is a light-cone projection the massive momentum K_{14} . The momenta q and \tilde{q} are related by conjugation.

A.2. The box coefficients: I

In this part we find the coefficients of two-mass-hard box functions where both massless legs $(i-2)$ and $(i-1)$ have equal (positive) helicities.

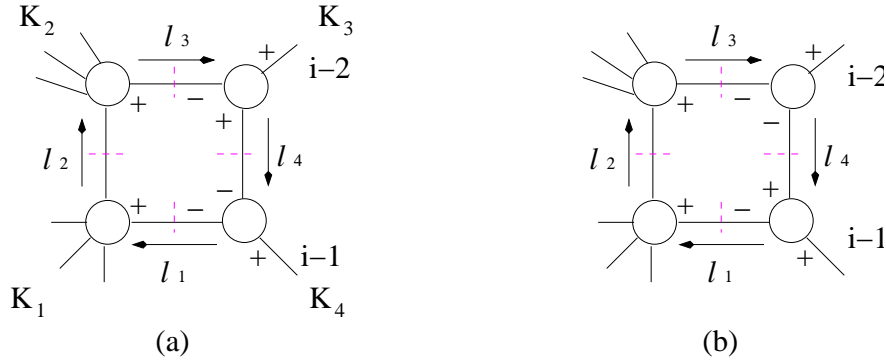


Fig. 5: The two possible helicity configurations of box functions where both massless legs $(i-2)$ and $(i-1)$ have equal positive helicities.

With both massless legs $(i-2)$ and $(i-1)$ having positive helicities, there are two possible contributions with different helicity assignments (see fig. 5). Since we have solved the loop momenta $\ell_1, \ell_2, \ell_3, \ell_4$ in the previous subsection, we can use the general formula from the quadruple cut to read off the corresponding coefficients.

As given in [19], the coefficient of the box integral $I^{2m\ h}$ is calculated by

$$\hat{d}_\alpha = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}. \quad (\text{A.7})$$

The coefficient of $F^{2m\ h}$ is then given by $d_\alpha = -2\hat{d}_\alpha/(K_{34}^2 K_{14}^2)$.

Consider first the configuration (a) of fig. 5. Since in this situation only gluons may circulate in the loop, we find that

$$\hat{d}_a = \frac{A_{\text{tree}}(K_4) A_{\text{tree}}(K_1) A_{\text{tree}}(K_2) A_{\text{tree}}(K_3)}{2}.$$

We use the solution (A.3) to determine that

$$A_{\text{tree}}(K_3) = \gamma[i-2\ i-1], \quad A_{\text{tree}}(K_4) = -\frac{\langle i-2\ i-1 \rangle}{\gamma}.$$

Since $K_{34}^2 = \langle i-2\ i-1 \rangle[i-2\ i-1]$, we find the result

$$d_a = \frac{A_{\text{tree}}(-\ell_1^+, K_1, +\ell_2^\pm) A_{\text{tree}}(-\ell_2^\mp, K_2, +\ell_3^+)}{K_{14}^2}, \quad (\text{A.8})$$

where the loop momenta are given by (A.3) and (A.4), i.e.

$$-\ell_1 = p_{i-1} - \delta p, \quad \ell_3 = p_{i-2} + \delta p, \quad \ell_2 = -K_{14} + \delta p, \quad \delta p = \frac{K_{14}^2}{2q \cdot K_{14}} q \quad (\text{A.9})$$

with $q_{\alpha\dot{\alpha}} = (\lambda_{i-2})_\alpha (\tilde{\lambda}_{i-1})_{\dot{\alpha}}$.

For configuration (b) of fig. 5 we do a similar calculation and get the coefficient

$$d_b = \frac{A_{\text{tree}}(-\ell_1^+, K_1, +\ell_2^\pm) A_{\text{tree}}(-\ell_2^\mp, K_2, +\ell_3^+)}{K_{14}^2} \quad (\text{A.10})$$

where the loop momenta are given by (A.5) and (A.6), i.e.,

$$-\ell_1 = p_{i-1} - \delta \tilde{p}, \quad \ell_3 = p_{i-2} + \delta \tilde{p}, \quad \ell_2 = -K_{14} + \delta \tilde{p}, \quad \delta \tilde{p} = \frac{K_{14}^2}{2\tilde{q} \cdot K_{14}} \tilde{q} \quad (\text{A.11})$$

with $\tilde{q}_{\alpha\dot{\alpha}} = (\lambda_{i-1})_\alpha (\tilde{\lambda}_{i-2})_{\dot{\alpha}}$.

The overall coefficient is the sum of the two contributions in (A.8) and (A.10).

A.3. The box coefficients: II

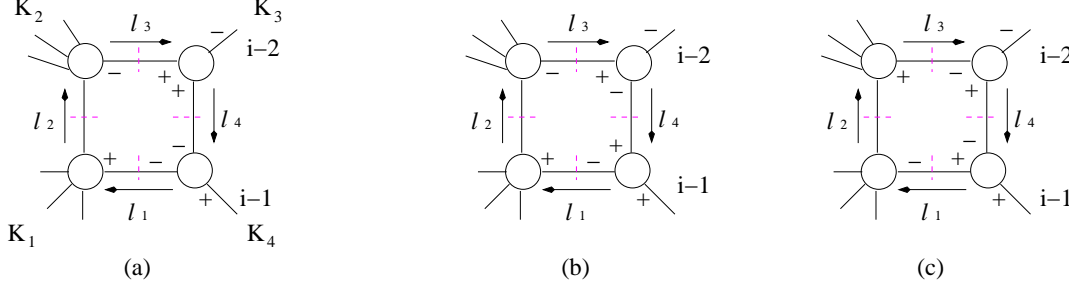


Fig. 6: Three possible helicity configurations where part (a) can only have gluons circulating in the loop while parts (b) and (c) can have fermions and complex scalars circulating.

In this subsection, we present the coefficients of two-mass-hard box functions where massless leg $(i - 2)$ has negative helicity and massless leg $(i - 1)$ has positive helicity. For this assignment of helicities there are three possible configurations (see fig. 6). The calculations are similar to the previous case, so we will be brief and mention only the new features.

For configuration (a), only gluons are allowed to circulate in the loop. With the exact same calculation we reach

$$d_a = \frac{A_{\text{tree}}(-\ell_1^+, K_1, +\ell_2^\pm) A_{\text{tree}}(-\ell_2^\mp, K_2, +\ell_3^-)}{K_{14}^2}, \quad (\text{A.12})$$

where the loop momenta are given by (A.3) and (A.4).

For configuration (b), depending on the helicity distribution at the vertices K_1 and K_2 , it is possible that chiral fermions and complex scalars of $\mathcal{N} = 4$ vector multiplet circulate in the loop. Thus we get the contribution of part (b) with helicity solution given by (A.5) and (A.6) as

$$d_b = \sum_{a=0,1,2} n_a \frac{A_{\text{tree}}^a(-\ell_1^+, K_1, +\ell_2^\pm) A_{\text{tree}}^a(-\ell_2^\mp, K_2, +\ell_3^-)}{K_{14}^2} \alpha^a, \quad (\text{A.13})$$

where $a = 0, 1, 2$ are for gluons, fermions and scalars and n_a are 1, -4 and 3 respectively. For example, when $a = 1$, the particles propagating with momenta ℓ_1 , ℓ_2 and ℓ_3 are all fermions.

Configuration (c) is similar to configuration (b). With the helicity solution given by (A.5) and (A.6) we find the contribution to coefficients from part (c) to be

$$d_c = \sum_{a=0,1,2} n_a \frac{A_{\text{tree}}^a(-\ell_1^-, K_1, +\ell_2^\pm) A_{\text{tree}}^a(-\ell_2^\mp, K_2, +\ell_3^+)}{K_{14}^2} \alpha^{4-a}. \quad (\text{A.14})$$

The overall coefficient is given by the sum of (A.12), (A.13) and (A.14).

As mentioned in the main text, it is the formula (A.12) that inspired us to make the conjecture (2.1). The relationship between the two formulas (A.12) and (2.1) is the following: $-\ell_1 \rightarrow \widehat{n}$, $\ell_2 \rightarrow -\widehat{P}_{n,i}$ and $\ell_3 \rightarrow \widehat{n-1}$. In principle, we can use formula (A.8) or (A.10) to calculate the tree level amplitude if we take the reference gluons to be $(i-2)^+$ and $(i-1)^+$. The reader can also find the expressions for coefficients of box functions with helicities $(i-2)^- (i-1)^-$ or $(i-2)^+ (i-1)^-$, which are related to the above two solutions by conjugation.

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